Fatigue:
Total Life Approaches
Fatigue Design Approaches

Long before the LEFM-based approaches (e.g. Paris Law, 1963) to characterize fatigue failure were developed, the importance of cyclic loading in causing failures (e.g. railroad axles) was recognized. Starting with the work of A. Wöhler (1860), who did rotating bend tests on various alloys, *empirical* methods have been developed. In this lecture we present two empirically-based design approaches, the *stress-life* approach and the *strain-life* approach.
Cyclic Loading

A typical stress history during cyclic loading is depicted below.
What are the important parameters to characterize a given cyclic loading history?

- **Stress Range:** \( \Delta \sigma = \sigma_{\text{max}} - \sigma_{\text{min}} \)

- **Stress amplitude:** \( \sigma_a = \frac{1}{2} \left( \sigma_{\text{max}} - \sigma_{\text{min}} \right) \)

- **Mean stress:** \( \sigma_m = \frac{1}{2} \left( \sigma_{\text{max}} + \sigma_{\text{min}} \right) \)

- **Load ratio:** \( R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} \)
• **Frequency**: $v$ or $f$ in units of Hz. For rotating machinery at 3000 rpm, $f = 50$ Hz. In general only influences fatigue if there are environmental effects present, such as humidity or elevated temperatures.

• **Waveform**: Is the stress history a sine wave, square wave, or some other waveform? As with frequency, generally only influences fatigue if there are environmental effects.
If a plot is made of the applied stress amplitude versus the number of reversals to failure (S-N curve) the following behavior is typically observed:
If the stress is below $\sigma_e$ (the *endurance limit* or *fatigue limit*), the component has effectively *infinite* life.

$$\sigma_e \approx 0.35\sigma_{TS} - 0.50\sigma_{TS}$$ for most steels and copper alloys.

If the material does not have a well defined $\sigma_e$, often $\sigma_e$ is arbitrarily defined as the stress that gives $N_f = 10^7$. 
Fatigue Design Approaches

If a plot is prepared of log($\sigma_a$) versus log (2$N_f$) (where 2$N_f$ represents the number of reversals to failure, one cycle equals two reversals) a *linear* relationship is commonly observed. The following relationship between stress amplitude and lifetime (Basquin, 1910) has been proposed:

$$\frac{\Delta \sigma}{2} = \sigma_a = \sigma'_f \left(2N_f\right)^b$$
In the previous expression $\sigma_f'$ is the fatigue strength coefficient (for most metals $\approx \sigma_f$, the true fracture strength), $b$ is the fatigue strength exponent or Basquin’s exponent ($\approx -0.05$ to $-0.12$), and $2N_f$ is the number of reversals to failure.
The total fatigue life of a component can be considered to have two parts, the *initiation life* and the *propagation life*, as depicted below.
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The preceding approach to calculate the lifetime assumes fully reversed fatigue loads, so that the mean stress $\sigma_m$ is zero. How do we handle cases where $\sigma_m \neq 0$?
\[
\sigma_a = \sigma_a \bigg|_{\sigma_m = 0} \left\{ 1 - \frac{\sigma_m}{\sigma_y} \right\} \quad (\text{Soderberg})
\]

\[
\sigma_a = \sigma_a \bigg|_{\sigma_m = 0} \left\{ 1 - \frac{\sigma_m}{\sigma_{TS}} \right\} \quad (\text{Goodman})
\]

\[
\sigma_a = \sigma_a \bigg|_{\sigma_m = 0} \left\{ 1 - \left( \frac{\sigma_m}{\sigma_{TS}} \right)^2 \right\} \quad (\text{Gerber})
\]
In the previous expressions, $\sigma_a$ is the stress amplitude denoting the fatigue strength for a nonzero mean stress, $\sigma_a\big|_{\sigma_m=0}$ is the stress amplitude (for a fixed life) for fully reversed loading ($\sigma_m = 0$ and $R = -1$), and $\sigma_y$ and $\sigma_{TS}$ are the yield strength and the tensile strength, respectively.

Soderberg gives a conservative estimate of the fatigue life.

Gerber gives good predictions for ductile alloys for tensile mean stresses. Note that it cannot distinguish between tensile and compressive mean stresses.
Fatigue Design Approaches

How do we handle situations where we have varying amplitude loads, as depicted below?

Different Amplitudes
The component is assumed to fail when the total damage becomes equal to 1, or

$$\sum_{i} \frac{n_i}{N_{fi}} = 1$$

It is assumed that the sequence in which the loads are applied has no influence on the lifetime of the component. In fact, the sequence of loads can have a large influence on the lifetime of the component.
A very common approach is the *Palmgren-Miner* linear damage summation rule.

If we define $2N_{fi}$ as the number of reversals to failure at $\sigma_{ai}$, then the partial damage $d$ for each different loading $\sigma_{ai}$ is

$$d = \frac{2n_i}{2N_{fi}} = \frac{\text{Reversals at } \sigma_{ai}}{\text{Reversals to failure at } \sigma_{ai}}$$
Consider a sequence of two different cyclic loads, $\sigma_{a1}$ and $\sigma_{a2}$. Let $\sigma_{a1} > \sigma_{a2}$.

**Case 1:** Apply $\sigma_{a1}$ then $\sigma_{a2}$.

In this case, $\sum_i \frac{n_i}{N_{f,i}}$ can be less than 1. During the first loading ($\sigma_{a1}$) numerous microcracks can be initiated, which can be further propagated by the second loading ($\sigma_{a2}$).
Case 2: Apply $\sigma_{a2}$ then $\sigma_{a1}$.

In this case $\sum_i \frac{n_i}{N_{f,i}}$ can be greater than 1. The first loading ($\sigma_{a2}$) is not high enough to cause any microcracks, but it is high enough to strain harden the material. Then in the second loading ($\sigma_{a1}$), since the material has been hardened it is more difficult to initiate any damage in the material.
Fatigue Design Approaches

The *stress-life* approach just described is applicable for situations involving primarily elastic deformation. Under these conditions the component is expected to have a long lifetime. For situations involving high stresses, high temperatures, or stress concentrations such as notches, where significant plasticity can be involved, the approach is not appropriate. How do we handle these situations?
Rather than the stress amplitude $\sigma_a$, the loading is characterized by the plastic strain amplitude $\Delta \varepsilon_p / 2$. Under these conditions, if a plot is made of $\log \left( \frac{\Delta \varepsilon_p}{2} \right)$ versus $\log(2N_f)$, the following linear behavior is generally observed:
To represent this behavior, the following relationship (Coffin-Manson, 1955) has been proposed:

\[
\Delta \varepsilon_p \frac{2}{2} = \varepsilon'_f \left(2N_f\right)^c
\]

where \( \Delta \varepsilon_p \) is the plastic strain amplitude, \( \varepsilon'_f \) is the fatigue ductility coefficient (for most metals \( \approx \varepsilon'_f \), the true fracture ductility) and \( c \) is the fatigue ductility exponent (-0.5 to -0.7 for many metals).
Fatigue Design Approaches

In general, how does one know which equation to apply (stress-life or strain-life approach)? Consider a fully reversed, strain-controlled loading. The total strain is composed of an elastic and plastic part, i.e.

\[ \frac{\Delta \varepsilon}{2} = \frac{\Delta \varepsilon_e}{2} + \frac{\Delta \varepsilon_p}{2} \]

The Coffin-Manson expression may used to express the term \( \Delta \varepsilon_p / 2 \). What about \( \Delta \varepsilon_e / 2 \)?
From the Basquin Law (stress-life approach):

\[ \frac{\Delta \sigma}{2} = \sigma'_f \left( 2N_f \right)^b \]

For 1-D, elastic loading and thus

\[ \frac{\Delta \varepsilon_e}{2} = \frac{\Delta \sigma}{2E} = \frac{\sigma_a}{E} \]

and thus

\[ \frac{\Delta \varepsilon_e}{2} = \frac{\sigma'_f}{E} \left( 2N_f \right)^b \]

and we may write that:

\[ \frac{\Delta \varepsilon}{2} = \frac{\sigma'_f}{E} \left( 2N_f \right)^b + \varepsilon'_f \left( 2N_f \right)^c \]
Fatigue Design Approaches

A plot of this is depicted below:
Fatigue Design Approaches

If the amplitude of the total strain is such that we have significant plasticity, the lifetime is likely to be short (Low Cycle Fatigue or LCF; strain life approach). If the stresses are low enough that the strains are elastic, the lifetime is likely to be long (High Cycle Fatigue or HCF; stress-life approach).

The transition life (at $2N_t$) is found by setting the plastic strain amplitude equal to the elastic strain amplitude.

$$\frac{\sigma'_f}{E}(2N_t)^b = \varepsilon'_f (2N_t)^c$$
At long component lifetimes ($N >> N_t$), the elastic strain is more dominant and *strength* will control performance. At short component lifetimes ($N << N_t$), plastic strain dominates, and *ductility* will control performance. Unfortunately, in most materials improvements in strength lead to reductions in ductility, and vice-versa.
Important: These total life approaches represent crack initiation life in smooth specimens. However, engineering materials contain inherent defects. Therefore, these approaches can lead to overestimation of useful life.
Fatigue Crack Inhibition

Shot peening is a cold working process in which the surface of a part is bombarded with small spherical media called *shot*. Each piece of shot striking the surface acts as a tiny peening hammer, imparting to the surface a small indentation or dimple. The net result is a layer of material in a state of residual compression. It is well established that cracks will not initiate or propagate in a compressively stressed zone.
Since nearly all fatigue and stress corrosion failures originate at the surface of a part, compressive stresses induced by shot peening provide \textit{considerable} increases in part life. Typically the residual stress produced is at least half the yield strength of the material being peened.

The benefits of shot peening are a result of the \textit{residual compressive stress} and the \textit{cold working} of the surface.
**Residual stress:** Increases resistance to fatigue crack growth, corrosion fatigue, stress corrosion cracking, hydrogen assisted cracking, fretting, galling and erosion caused by cavitation.

**Cold Working:** Benefits include work hardening (strengthening), intergranular corrosion resistance, surface texturing, closing of porosity and testing the bond of coatings.
Fatigue Crack Inhibition

• Blades, buckets, disks and shafts for aircraft jet engines; (e.g., blade roots are peened to prevent fretting, galling and fatigue).
• Crank shafts used in ground vehicles.
• Shot peening of plasma-spray-coated components before and after spraying.
• Shot peening of gears used in automotive and heavy vehicle components, marine transmissions, small power tools and large mining equipment.
• Compression coil springs in automobiles.
• Peening and peen forming of wing skin for aircraft.
Residual stresses are those stresses remaining in a part after all manufacturing operations are completed, and with no external load applied. In most applications for shot peening, the benefit obtained is the direct result of the residual compressive stress produced.
A typical residual stress profile created by shot peening is shown below:
The beneficial effect of shot peening in improving the endurance limit is depicted below:
Fatigue Crack Growth
Fatigue Crack Growth

**Key Idea:** Fluctuating loads are more dangerous than monotonic loads.

**Example:** *Comet Airliner* (case study). The actual cabin pressure differential when the plane was in flight was $\approx 8.5$ pounds per square inch (psi). The design pressure was $\approx 20$ psi (a factor of safety greater than 2). Thought to be safe! However, crack growth due to *cyclic* loading caused catastrophic failure of the aircraft.
Cyclic Loading

A typical stress history during cyclic loading is depicted below:
What are the important parameters to characterize a given cyclic loading history?

- **Stress Range**: \( \Delta \sigma = \sigma_{\text{max}} - \sigma_{\text{min}} \)

- **K-Range**: \( \Delta K = K_{\text{max}} - K_{\text{min}} \)

- **Stress amplitude**: \( \sigma_a = \frac{1}{2}(\sigma_{\text{max}} - \sigma_{\text{min}}) \)

- **Mean stress**: \( \sigma_m = \frac{1}{2}(\sigma_{\text{max}} + \sigma_{\text{min}}) \)

- **Load ratio**: \( R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} = \frac{K_{\text{min}}}{K_{\text{max}}} \)
• **Frequency:** \( \nu \) or \( f \) in units of Hz. For rotating machinery at 3000 rpm, \( f = 50 \) Hz. In general only influences fatigue crack growth if there are environmental effects present, such as humidity or elevated temperatures.

• **Waveform:** Is the stress history a sine wave, square wave, or some other waveform? As with frequency, generally only influences fatigue crack growth if there are environmental effects.
Cyclic vs Static Loading

Key difference between *static* and *cyclic* loading:

**Static:** Until applied $K$ reaches $K_c$ (30 MPa $\sqrt{m}$ for example) the crack will not grow.

**Cyclic:** $K$ applied can be well below $K_c$ (3 MPa $\sqrt{m}$ for example). Over time, the crack grows.

The design may be safe considering static loads, but any cyclic loads **must** also be considered.
Fatigue Crack Growth

In 1963 the ideas of LEFM were applied to fatigue crack growth by Paris, Gomez and Anderson.

For a given cyclic loading, define $\Delta K$ as $K_{\text{max}} - K_{\text{min}}$ which can be found from $\Delta \sigma$ and the geometry of the cracked body.

Say that the crack grows an amount $\Delta a$ during $N$ cycles. Paris, Gomez and Anderson said that the rate of crack growth depends on $\Delta K$ in the following way:

$$\frac{\Delta a}{\Delta N} \rightarrow \frac{da}{dN} = C(\Delta K)^m$$
Thus a plot of log (\(da/dN\)) vs. log (\(\Delta K\)) should be a straight line with a slope of \(m\).

The actual relationship between crack growth rate and \(\Delta K\) is depicted on the following page. There are three different regimes of fatigue crack growth, A, B and C.
The graph shows the relationship between crack growth rate and stress intensity factor ($\Delta K$) in terms of $da/dN$ (mm/cycle) on the y-axis and $\log \Delta K$ on the x-axis. It is divided into three regimes:

- **Regime A**: Crack growth rate is slow and not dependent on $\Delta K$.
- **Regime B**: Crack growth rate increases with $\Delta K$.
- **Regime C**: Crack growth rate is rapid and linear with $\Delta K$.

The crack growth rate at 50 Hz is indicated by the dashed line and includes the following rates:

- 1 mm/min
- 1 mm/hour
- 1 mm/day
- 1 mm/week

The equation $da/dN = C(\Delta K)^m$ is used to describe the crack growth rate in regime C, where $C$ and $m$ are constants.
## Fatigue Crack Growth

### Three Regimes

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<td>$r_c \geq d_g$</td>
<td>$r_c \gg d_g$</td>
</tr>
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</table>

* large influence on crack growth for certain combinations of environment, load ratio and frequency.  
† $r_c$ and $d_g$ refer to the cyclic plastic zone size and the grain size, respectively.
Fatigue Crack Growth

Regime A

Concept of the threshold stress intensity factor $\Delta K_{th}$:

when $\Delta K \approx \Delta K_{th}$, where $\Delta K_{th}$ is the threshold stress intensity factor, the rate of crack growth is so slow that the crack is often assumed to be dormant or growing at an undetectable rate.

An operational definition for $\Delta K_{th}$ often used is that if the rate of crack growth is $10^{-8}$ mm/cycle or less the conditions are assumed to be at or below $\Delta K_{th}$. 
An important point is that these extremely slow crack growth rates represent an average crack advance of less than one atomic spacing per cycle. How is this possible? What actually occurs is that there are many cycles with no crack advance, then the crack advances by 1 atomic spacing in a single cycle, which is followed again by many cycles with no crack advance.
When we are in regime B (Paris regime) the following calculation can be carried out to determine the number of cycles to failure. From the Paris Law:

\[
\frac{da}{dN} = C(\Delta K)^m
\]

\(\Delta K\) can be expressed in terms of \(\Delta \sigma\)

\[
\Delta K = Y\Delta \sigma \sqrt{\pi a}
\]

Where \(Y\) depends the specific specimen geometry.
Thus the Paris Law becomes:

$$\frac{da}{dN} = C \left( Y \Delta \sigma \sqrt{\pi a} \right)^m$$

Assume that $Y$ is a constant. Integrate both sides:

$$\int_{a_0}^{a_f} \frac{da}{a^{m/2}} = CY^m \left( \Delta \sigma \right)^m \pi^{m/2} \int_0^{N_f} dN$$
For $m > 2$:

$$N_f = \frac{2}{(m-2)CY^m(\Delta \sigma)^m \pi^{m/2}} \left[ \frac{1}{(a_0)^{(m-2)/2}} - \frac{1}{(a_f)^{(m-2)/2}} \right]$$

For $m = 2$:

$$N_f = \frac{1}{CY^2(\Delta \sigma)^2 \pi} \ln \frac{a_f}{a_0}$$
The constants $C$ and $m$ are material parameters that must be determined experimentally. Typically $m$ is in the range 2-4 for metals, and 4-100 for ceramics and polymers.

For cases where $Y$ depends on crack length, these integrations generally will be performed numerically.

**Important:** Note that in the Paris regime the rate of crack growth is weakly sensitive to the load ratio $R$. The key parameter governing crack growth is $\Delta K$. 
In these expressions, we need to determine the initial crack length $a_0$ and the final crack length $a_f$ (sometimes called the critical crack length).

How do we determine the initial crack length $a_0$?

Cracks can be detected using a variety of techniques, ranging from simple visual inspection to more sophisticated techniques based ultrasonics or x-rays. If no cracks are detectable by our inspection, we must assume that a crack just at the resolution of our detection system exists.
How do we determine the final crack length $a_f$? We know that eventually the crack can grow to a length at which the material fails immediately, i.e.

$$K_{\text{max}} \rightarrow K_c$$

or

$$Y\sigma_{\text{max}} \sqrt{\pi a_f} \rightarrow K_c$$
Thus we may solve for $a_f$ as follows:

$$a_f = \frac{1}{\pi} \frac{K_c^2}{Y_a^2 \sigma_{\text{max}}^2}$$

A very important idea that comes from this analysis is the following: even if a component has a detectable crack, it need not be removed from service! Using this framework, the remaining life can be assessed. The component can remain in service provided it is inspected periodically. This is the crack-tolerant or damage tolerant design approach.
Fatigue Striations

An advancing fatigue crack leaves characteristic markings called *striations* in its wake. These can provide evidence that a given failure was caused by fatigue. The striations on the fracture surface are produced as the crack advances over one cycle, i.e. each striation corresponds to $da$. 
Fatigue striations on the fracture surface (2024-T3 aluminum). Crack advance was from the lower left to the upper right.
Models for Paris Regime

1. Geometric (or) CTOD models

\[
\frac{da}{dN} \approx \Delta \delta_t = \beta \frac{(\Delta K)^2}{\sigma'_y E'}
\]

\approx \text{striation spacing}

Important: This implies that the Paris Exponent \( m \) is equal to 2.
Advantages of CTOD-based models:

• Physically appealing
• Usefulness in multiaxial fatigue
• Link to microstructural size

2. Damage Accumulation Models

Basic Idea: Accumulation of local plastic strain over some microstructurally significant distance at crack tip leads to fracture.

\[
\frac{da}{dN} = \frac{4\sigma}{u^*} \int_0^{rc} \Delta u_y (x,0) dx
\]
In the previous expression $u^*$ represents the critical hysteresis energy, or the “work of fracture” and $r_c$ represents some microstructurally significant distance.

\[
\frac{da}{dN} \propto \frac{(\Delta K)^4}{\mu \sigma_y^2 u^*}
\]

Note: This implies that the Paris Exponent $m$ is equal to 4.

(Integration of local $\sigma - \varepsilon$ hysteresis loops). Strain gradients at crack tip.
Fatigue Crack Growth

a) Near-Threshold: \( r_y < d \)
   (Stage I, Modes II and I)

b) Higher Growth Rates: \( r_y > d \)
   (Stage II, Mode I)

(Forsyth, 1962)
Fatigue Crack Growth

Stage I
Fatigue Striations

Stage II

slip bands

fatigue crack
Fatigue Striations
Irreversible slip (oxidation)

two cycles

Reversed slip (vacuum)

two cycles

loading and unloading
Fatigue Design

• Determine typical service spectra.

• Estimate useful fatigue life based on laboratory tests or analyses.

• Add factor of safety.

• At the end of the expected life, the component is retired from service, even if no failure has occurred and component has considerable residual life.

• Emphasis on prevention of crack initiation.
• Approach is theoretical in nature.

• Even if an individual member of a component fails, there should be sufficient **structural integrity** to operate safely.

• Multiple load paths and **crack arresters**.

• Mandates periodic inspection.

• Accent on crack growth.
Retirement For Cause

F-100 gas turbine engines in F-15 and F-16 fighter aircraft for U.S. Air force.

Old Approach:
• 1000 disks could be retired from service when, statistically, only one disk had a fatigue crack \((a \leq 0.75 \text{ mm})\).

New, RFC Approach (since 1986):
• Retirement of a component occurs when the unique fatigue life of that particular component is expended.
• Retirement only when there is reason for removal (e.g. crack).

• Twin engine F-15 and single engine F-16.

• 3200 engines in the operational inventory of U.S. Air Force.

• 23 components of the engine are managed under RFC.

• 1986-2005 life cycle cost savings: $1,000,000,000

• Additional labor and fuel costs: $655,000,000
Fracture and Fatigue
Examples
Fatigue cracking under low-amplitude, high-cycle loading is the dominant failure process in a number of engineering applications. In this case study we examine the origins and effects of high-cycle fatigue in advanced gas turbine jets used in military aircraft. The principal cause of failure of components, from which a gas turbine jet engine of a modern military aircraft is made, is high-cycle fatigue. As illustrated in the following figure, fatigue failure accounts for 49% of all component damage in jet engines.
High-cycle fatigue (HCF) is responsible for nearly half these failures, whilst low-cycle fatigue (LCF) and all other modes of fatigue lead to the remainder of fatigue failures in roughly equal proportions.
Failure by HCF affects a variety of engine components, as shown in the figure below.
The origin of HCF in the gas turbine aircraft engine can be attributed to one or more of the following causes:

1. Mechanical vibration arising from rotor imbalance (which affects plumbing, nonrotating structures and external members) and rub (which affects blade tips and gas path seals).

2. Aerodynamic excitation occurring in upstream vanes, downstream struts and blades, whereby engine excitation frequencies and component response frequencies corresponding to different modes of vibration may overlap.
3. Aeromechanical instability, primarily in blades, accompanying aerofoil flutter.

4. Acoustic fatigue of sheet metal components in the combustor, nozzle and augmentor.
The above sources of low amplitude, HCF are augmented by the following damage processes which create microscopic notches and other sites at which fatigue cracks can nucleate and advance subcritically to catastrophic proportions,

1. **Foreign Object Damage (FOD)**, which usually occurs in compressor and fan blades: FOD can induce micronotches, tears, dents and gouges that may vary in dimensions from tens of micrometers to tens of millimeters, depending on the size, nature and severity of the impact of the foreign object. Sources of FOD are as diverse as sand particles...
and birds. As noted earlier, the edge of the fan blade, which is susceptible to FOD, is laser shock peened for improved fatigue resistance.

2. **Domestic Object Damage (DOD)**, which arises from a dislodged debris or component from another location of the engine.

3. **Fretting Fatigue**, which occurs at blade and disk attachment surfaces (dove-tail or fir tree section), bolt flanges, and shrink fit areas.

4. **Galling**, which occurs in the same regions as fretting, expect that it involves greater displacements due to major engine throttle and speed changes.
5. **HCF-LCF interactions**, where HCF is exacerbated by LCF as, for example when creep and thermo-mechanical fatigue in hot sections (such as turbine blades) cause further reductions in fatigue life, over and above that due to vibrations.

Current methods to assess the HCF life of critical components in aircraft gas turbine engines entail the following general steps.

- Appropriate stress analysis (largely based on the finite-element method) are performed to determine the mean stress level.
- Structural dynamics simulations are carried out to determine resonant frequencies and excitation modes.
• The design of the component is then carried out in such a way that
  - it meets the criteria for safe life for HCF with the appropriate mean stress level (using the modified Goodman diagram), and
  - no resonance-related problems arise.

The parameters that serve as input to design are gathered from specimen testing and component testing, and the stress-life approach is empirically modified to allow for reductions in life due to FOD, DOD, fretting and galling.
Example Problem

Energy Release Rate

Consider the cracked elastic plate shown on the following page, with Young’s modulus $E$ and Poisson’s ratio $\nu$. The height $H$ is small compared to the dimensions $a$ and $b$, and the thickness $B$ is small enough that plane stress conditions ($\sigma_{zz} = 0$) may be assumed.
Imagine that the bottom of the specimen is held fixed, while the top is extended by an amount $\delta$ in the $y$-direction, with $\delta << H$, so that an axial strain $\delta / H$ is imposed. Assume that the material above and below the crack is free of stress. Further assume that the material to the right of the crack has a constant stress $\sigma_{yy} = \sigma$ corresponding to the imposed axial strain. Since the dimension $b$ is much larger than $H$ you should assume that $\varepsilon_{yy} = 0$. 
Calculate the total strain energy $\Phi$ in the plate where

$$\Phi = \int_\nu \phi \, d\nu = \frac{1}{2} \int_\nu \sigma_{ij} \varepsilon_{ij} \, d\nu$$

Your answer should be in terms of $B$, $H$, $b$, $\delta$ and $E$ and $\nu$.

**Solution**

The strain energy density $\phi$ is given as $1/2(\sigma_{ij} \varepsilon_{ij})$, i.e $1/2(\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + etc)$. The only nonzero term in this expansion will be $1/2(\sigma_{yy} \varepsilon_{yy})$. We know that $\varepsilon_{yy} = \delta/H$, so we need to solve for $\sigma_{yy}$. We know that
\[
\varepsilon_{yy} = \frac{1}{E} \left( \sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz}) \right)
\]

We also know that \(\sigma_{zz} = 0\) (plane stress). We have also assumed that \(\varepsilon_{xx} = 0\), so that

\[
\varepsilon_{xx} = 0 = \frac{1}{E} \left( \sigma_{xx} - \nu \sigma_{yy} \right)
\]

and thus

\[
\sigma_{xx} = \nu \sigma_{yy}
\]
so

\[\varepsilon_{yy} = \frac{1}{E} \sigma_{yy} \left(1 - \nu^2\right)\]

and finally

\[
\sigma_{yy} = \frac{E}{1 - \nu^2} \varepsilon_{yy} = \frac{E}{1 - \nu^2} \frac{\delta}{H}
\]

so

\[\Phi = \frac{1}{2} \int_{\nu} \phi d\nu = \frac{BbHE}{2(1 - \nu^2)} \left(\frac{\delta}{H}\right)^2\]
If the crack grows all the way across the plate and the plate separates into two pieces, the system will have zero strain energy. So the strain energy decreases as the crack grows. Calculate the change in strain energy $d\Phi$ of the plate if the crack grows by some amount $da$. Use the same assumptions used previously to calculate the total strain energy. Then use the result that

$$
\zeta = -\frac{1}{B} \left[ \frac{\partial \Phi}{\partial a} \right]_{u \ fixed}
$$
to evaluate the energy release rate $\mathcal{G}$ for this geometry. Assuming that plane stress conditions hold, convert this $\mathcal{G}$ to a stress intensity factor $K$.

**Solution**

Following the same assumptions above, if the crack grows by an amount $da$ we lose an amount of strain energy equal to $\phi$ (calculated above) times the volume $BHda$ (volume that previously contained strain energy but is now unstressed, due to crack growth). So that
\[ d\Phi = -\frac{BHE}{2(1-v^2)} \left( \frac{\delta}{H} \right)^2 \, da \]

and thus

\[ \zeta = -\frac{1}{B} \left[ \frac{\partial \Phi}{\partial a} \right]_{u \text{ fixed}} = \frac{HE}{2(1-v^2)} \left( \frac{\delta}{H} \right)^2 \]

Use the relationship between energy release rate and stress intensity factor for plane stress:
Assume that this material has a known fracture toughness (plane stress) $K_{Ic}$. Calculate the value of $\delta$ that must be imposed to make the crack advance.
Solution

To find the critical value of $\delta$ for crack advance, simply set $K$ (determine above) equal to $K_{Ic}$ and solve for $\delta$:

$$\delta = \frac{K_{Ic}}{E} \sqrt{2H(1-\nu^2)}$$
The interface toughness of an adhesive is to be estimated on the basis of a proposed method which entails the bonding of a thin elastic disk with the adhesive on a rigid substrate. Pressure is applied to the bonded side of the disk through a tiny hole in the substrate, by pumping a fluid. This pressurization results in partial debonding of the disk and the formation of a ‘bulge’ or ‘blister’ in the disk. This method is hence referred to as the ‘pressurized bulge test’ or ‘blister test.’
Consider a two-dimensional version of this test which is shown in the figure. A thin beam is adhesively bonded to a rigid surface through which an incompressible fluid is forced resulting in a blister of half-width \( l \). Assuming that all concepts of linear elastic fracture mechanics may be extended to this case, apply the compliance method to determine the tensile ‘crack-tip’ stress intensity factor in terms of the blister dimension \( l \) and the piston displacement \( q \).
Assume that $q = 0$ when the beam is undeflected, and that all length dimensions normal to the plane of the figure are unity. Let the thickness of the beam and its elastic modulus be $h$ and $E$, respectively. You are given the following information:

1. The transverse deflection of an elastic double cantilever beam of length $2l$ subjected to a uniform pressure is

$$\omega(x) = \frac{p(l^2 - x^2)^2}{2Eh^3}$$
2. A measure of the compliance $c(l)$ may be defined as $q = c(l) Q$, where $Q$ is the force acting on the piston per unit thickness as shown in the figure.

3. 

$$\int_{-l}^{l} w(x) dx = qH$$

4. From the compliance interpretation of the energy release rate, $\zeta$, it is known that

$$\zeta = \frac{1}{2} \frac{dc}{dl} Q^2$$
Solution

The compliance of the system is defined as

\[ q = c(l)Q \]

Observe that (i) the pressurized fluid is incompressible, (ii) the pressure under the blister is equal to the pressure in the piston, and (iii) the pressure in the piston is directly related to \( Q \) by

\[ p = \frac{Q}{H} \]
Substitute the expression for the displacement $w$ into the incompressibility condition to obtain:

$$p = \frac{15}{8} \frac{EqHh^3}{l^5}$$

Use these two equations to eliminate $p$

$$q = \frac{8}{15} \frac{1}{E} \frac{l^5}{h^3 H^2} Q$$

This is our compliance relationship.
Using the compliance interpretation of $\zeta$, (note: there are two crack tips at the blister) we find that

$$c(l) = \frac{8}{15} \frac{1}{E} \frac{l^5}{h^3 H^2}$$

$$\zeta = \zeta_{\text{per tip}} = \frac{1}{2} \cdot \left( \frac{dc}{dl} Q^2 \right) = \frac{4}{6} \frac{1}{E} \frac{l^4}{h^3 H^2} Q^2$$

But $q = c(l)Q$. Using this result

$$\zeta = \frac{75}{32} E \frac{h^3 H^2}{l^6} q^2$$
For plane stress, we take this equation with the link between $K_I$ and $\varsigma$ to get

$$K_I = \left( \varsigma E \right)^{1/2} = \left( \frac{75}{32} E^2 \frac{h^3 H^2}{l^6} q^2 \right)^{1/2},$$

or

$$K_I = \frac{5}{4} \sqrt{\frac{3}{2}} \frac{Hh\sqrt{h}}{l^3} qE$$
Young’s modulus $E$ appears in this equation because $K_I$ has been written in terms of the displacement $q$. If we make use of the known relationship between $q$ and $Q$, the stress intensity factor becomes

$$K_I = \sqrt{\frac{2}{3}} \frac{l^2}{Hh\sqrt{h}} Q$$

which is the desired result.
Stability of Crack Growth

In this worked example, we discuss the driving force for fracture in brittle solids and the issues of stability of subcritical crack growth by recourse to the concepts of strain energy release rate which were discussed in lecture. Consider a cracked specimen which is loaded under compliant conditions in Mode I, as shown in the following figure. The specimen has unit thickness \( B = 1 \). \( C \) is the compliance (a function of crack length) of the specimen and \( C_M \) is the compliance of the testing machine (which may be regarded as a spring) in which the specimen is loaded in tension. The testing machine is in series with the specimen.
Let $\Delta_T$ be the total displacement which can be regarded as prescribed.

\[
\Delta_T = \Delta + C_M P = \Delta + \left( \frac{C_M}{C} \right) \Delta.
\]

The potential energy is given by

\[
W_p = \Phi + \frac{1}{2} C_M P^2 = \frac{1}{2} \frac{\Delta^2}{C} + \frac{1}{2} \frac{C_M}{C^2} \Delta^2
\]

\[
= \frac{1}{2} \frac{\Delta^2}{C} + \frac{1}{2} \left[ \frac{(\Delta_T - \Delta)^2}{C_M} \right]
\]
1. Show that $C_M = 0$ corresponds to fixed grip loading (constant displacement), and that $C_M \rightarrow \infty$ corresponds to dead-weight loading.

2. Show that the mechanical strain energy release rate,

$$\varsigma = -\left[ \frac{\partial(W_P)}{\partial a} \right]_{\Delta_T} = \frac{1}{2} P^2 \frac{dC}{da}$$

and therefore, does not depend on the compliance of the testing machine.
3. Show that \( \left[ \frac{\partial \varsigma}{\partial a} \right]_{\Delta T} \) depends on the nature of the loading system.

4. For an ideally brittle solid, show that crack advance is stable if \( \left[ \frac{\partial \varsigma}{\partial a} \right]_{\Delta T} \leq 0 \) and that it is unstable if \( \left[ \frac{\partial \varsigma}{\partial a} \right]_{\Delta T} > 0 \).
1. For fixed grip loading, $\Delta_T = \Delta$. From this, we note that $C_M/C = 0$ and $C_M = 0$. For dead-weight loading, the rate of change of the load $P$ with time is zero, i.e.

\[ \dot{P} = \frac{\dot{\Delta}}{C} = 0 \]

where $\dot{\Delta}$ is the rate of change of displacement.

Taking the derivative of our original expression for the total displacement we see that
\[
\dot{\Delta} = \frac{\dot{\Delta}_T}{1 + \left(\frac{C_M}{C}\right)} = 0.
\]

From this, we note that \(C_M \rightarrow \infty\) for dead weight loading.

2. Using the definition of the strain energy release rate and the fact that \(B = 1\), the mechanical strain energy release rate is

\[
\zeta = -\frac{dW_P}{da}
\]

From the previous expressions for \(\Delta_T\) and \(W_P\)

\[
W_P = \frac{\Delta_T^2}{2(C + C_M)}
\]
\[ \zeta = - \left[ \frac{\partial (W_P)}{\partial a} \right]_{\Delta T} = \frac{\Delta^2_T}{2 \left( C + C_M \right)^2} \cdot \frac{dC}{da} \]

Note that

\[ \frac{\Delta_T}{C_T} = \frac{\Delta_M}{C_M} = \frac{\Delta}{C} = P \]

\[ \Delta_T = \Delta_M + \Delta \]

From this equation, we note that \( C_M + C = C_T \).
Thus it is seen that
\[ \varsigma = \frac{P^2}{2} \cdot \frac{dC}{da} \]

Note that this expression for \( \varsigma \) does not involve the compliance of the testing machine.

3. From the previous result,

\[
\frac{\partial \varsigma}{\partial a} = \frac{\Delta_T^2}{(C + C_M)^3} \left( \frac{dC}{da} \right)^2 + \frac{1}{2} \frac{\Delta_T^2}{(C + C_M)^2} \left( \frac{d^2 C}{da^2} \right)
\]

\[
= -\frac{P^2}{(C + C_M)} \left( \frac{dC}{da} \right)^2 + \frac{P^2}{2} \left( \frac{d^2 C}{da^2} \right)
\]
4. For an ideally brittle solid, $d\zeta_R/da$ (change in resistance to static fracture with crack growth) is equal to zero. Thus, it is seen that crack advance is stable if $\left[ \frac{\partial \zeta}{\partial a} \right]_{\Delta T} \leq 0$ and unstable if $\left[ \frac{\partial \zeta}{\partial a} \right]_{\Delta T} \geq 0$. 
Fatigue Life Calculation

A circular cylindrical rod with a uniform cross-sectional area of 20 cm\(^2\) is subjected to a mean axial force of 120 kN. The fatigue strength of the material, \(\sigma_a = \sigma_{fs}\) is 250 MPa after 10\(^6\) cycles of fully reversed loading and \(\sigma_{TS} = 500\) MPa. Using the different procedures discussed in class, estimate the allowable amplitude of force for which the shaft should be designed to withstand at least one million fatigue cycles. State all your assumptions clearly.

**Solution**

The different expressions we have to assess the influence of mean stresses are:
In all cases $\sigma_a = \sigma_a \mid_{\sigma_m = 0} = 250 \text{ MPa}$ and $\sigma_m = (120,000 \text{ N/.0020 m}^2) = 60 \text{ MPa}$. The Modified Goodman and Gerber criteria can be applied directly to give:
Applying the Soderberg criterion requires a bit more thought since it includes the yield strength (which is not given) rather than the tensile strength. Depending on the details of the material behavior, the yield stress $\sigma_{YS}$ could be the same as the tensile strength $\sigma_{TS}$ (for very brittle materials) or as low as $\approx 0.5 \sigma_{TS}$ (for
very ductile materials). I will assume that 
$\sigma_{YS} = 0.7\sigma_{TS} = 350$ MPa. Thus the Soderberg criterion gives:

$$\sigma_a = 250 \left\{ 1 - \frac{60}{350} \right\} = 207.1 \text{ MPa}, \ (\text{Soderberg})$$

As you can see, you get significantly different answers depending on the model used. The Soderberg gives the most conservative result, while Gerber is the least conservative.