A METHOD FOR MEASURING RESIDUAL STRAINS IN FIBER-REINFORCED TITANIUM MATRIX COMPOSITES

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Abstract—A new method for measuring the residual fiber strains in fiber-reinforced titanium matrix composites has been developed. The method involves selectively etching the matrix over a prescribed length of composite and subsequently measuring the extension of the relaxed fibers relative to neighboring fibers that are still embedded within the matrix material. The extensions are measured using confocal microscopy. The method is demonstrated on three unidirectionally reinforced composites with varying fiber volume fractions. The effects of specimen tilt and fiber splaying following dissolution on the measured fiber extensions are analyzed. The residual fiber strains are rationalized on the basis of the thermoelastic properties of the constituents through a concentric cylinder model. The current method can be applied to other metal matrix composites reinforced with large diameter (monofilament) fibers, provided the matrix can be selectively etched.

1. INTRODUCTION

Significant levels of residual stress can exist in fiber-reinforced titanium matrix composites (TMCs) [1–3]. These stresses arise primarily from the mismatch in thermal expansion coefficients between the fibers and the matrix.† In Ti/SiC systems, the matrix thermal expansion coefficient, $\alpha_m$, exceeds that of the fibers, $\alpha_f$. Consequently, after cooling from the processing temperature, the matrix experiences an axial tensile stress whereas the fiber experiences an axial compressive stress. Moreover, the fiber-matrix interface experiences a normal compressive (clamping) stress.

The residual stresses have important effects on the mechanical properties of TMCs. For example, under tensile loading transverse to the fibers, the interfaces separate when the local normal stress (applied plus residual) just becomes tensile; the interface exhibits essentially zero normal strength [4–7]. In this context, the residual compression is desirable since it elevates the applied stress required for interface separation. Under tensile loading parallel to the fibers, the residual stresses have two opposing effects [8]. First, the residual tension in the matrix reduces the applied strain required for matrix yielding. Second, the axial compression in the fibers delays the onset of fiber fracture and thus enhances the failure strain of the composite.

Three types of experimental methods have been utilized to measure residual strains. The first of these involves measurement of peak shifts in the X-ray diffraction spectrum for the matrix [1, 9, 10]. These measurements are generally limited to near-surface regions (within ~10 $\mu$m). The second method is also based on diffraction, though with neutrons rather than X-rays [2, 11, 12]. Neutron diffraction has an advantage over X-ray diffraction in that neutron beams can penetrate through much thicker sections of materials (by factors of ~10$^3$) and thus the measured strains are representative of “bulk” material. The third method involves selectively etching the matrix from a central portion of a composite panel and subsequently measuring the change in length of the fibers resulting from the relaxation in the residual strain [1]. In principle, the stresses and strains within the matrix and the fibers can be calculated using the measurements from any of the three methods through an appropriate cell model.

The present article describes another method for measuring residual strains, based on a variation of the selective etching method. In this case, two long narrow slits are machined into the composite along the fiber length, with the matrix within the central portion being subsequently etched away and the length change of the fibers measured. The current method is relatively simple to perform and yields high precision in the strain measurement. The procedure is demonstrated using three unidirectional Ti/SiC composites. The measurements are analyzed using a concentric cylinder model. The potential advantages of the present method over the existing selective etching method are briefly discussed.

†The strains associated with phase transformations within the matrix (at ~1000°C) are accommodated almost completely by creep deformation within the matrix, causing minimal additional stress.
2. MATERIALS

Three unidirectionally-reinforced Ti/SiC composites were used in this study. The panels are designated A, B and C (Table 1). In all cases, the matrix was a Ti-6Al-4V alloy. Panel A was reinforced unidirectionally with 32% of SM-1240 (Sigma) SiC fibers, produced by British Petroleum. The panel was comprised of 6 plies with a total thickness of 1.0 mm [Fig. 1(a)]. Panels B and C were reinforced unidirectionally with 6 piles of SCS-6 SiC fibers, produced by Textron. These panels had been clad with monolithic Ti-6Al-4V. The cladding thicknesses were ~0.34 and 0.66 mm in panels B and C, respectively [Figs 1(b) and (c)]. The corresponding fiber volume fractions were 0.21 and 0.16.†

3. TEST METHOD

A schematic of the test specimen is shown in Fig. 2. The specimens were ~15 mm wide (transverse to the fibers) and ~75 mm long (parallel to the fibers). Two narrow slits, ~1 mm apart and ~50 mm long were machined along the length of the specimens using electrodischarge machining (EDM). The slit width was ~300 μm. The region outside of the narrow tongue between the slits was masked with epoxy. The exposed portion of the material within the slit [normal to the broad face sketched in Fig. 2(a)] was masked by inserting a strip of teflon tape into the slit and sealing it with epoxy. The entire specimen was then immersed in a 50% HF solution to dissolve the matrix between the slits, allowing the fibers to relax.

The relative extension, Δ, of the exposed fibers was measured using confocal microscopy. For this purpose, the specimen was inserted into the microscope with the fiber axis aligned with the beam direction. The heights, z, of the tops of the fibers, both in the exposed and the intact regions, were then measured. Typically, measurements were made on ~20-40 fibers embedded within the composite and on almost all of the exposed fibers (~20-30). The precision of an individual measurement, obtained by repeatedly measuring the height of an individual fiber, was ~1 μm. The positions, (x, y), of the fibers in the plane normal to the beam were also recorded. The axial residual fiber strain, $\varepsilon_i$, was calculated using the relation

$$\varepsilon_i = -\Delta / l, \quad (1)$$

where $l$ is the length of exposed fiber (~50 mm) and Δ is calculated from the difference in heights, $z$, between the exposed and embedded fibers. The corresponding stresses were calculated using a concentric cylinder model, detailed in Section 4.2.

4. MEASUREMENTS AND ANALYSIS

4.1. Residual fiber strain

Figure 3(a) shows the locations (x, y) of the measurements on Panel A. Within the embedded regions, measurements were made across the entire thickness (in the y-direction) and along a length ~10 mm normal to the through-thickness direction (x). Since the top surface of the panel could not be aligned perfectly normal to the beam axis, a slight but systematic variation in the heights of the embedded fibers was found along both the x- and y-directions. To account for this variation, the measurements on the embedded fibers were used to define a reference plane, corresponding to the top surface of the specimen prior to matrix dissolution. This was accomplished by performing a least squares fit of the data to the equation of a plane. For panel A, the slopes of the reference plane in the x- and y-directions were $\partial z / \partial x \approx 0.004$ and $\partial z / \partial y \approx 0.008$. The distribution in the differences between the measured heights and the ones calculated from the least squares fit are presented in Fig. 3(b). The standard deviation in this distribution is ~2 μm: comparable to the precision of the individual measurements.

In principle, the extension of each individual exposed fiber could be calculated from the differences between the height of the fiber following dissolution and the corresponding height in the reference plane at the location of the fiber prior to matrix dissolution. However, following dissolution, the fibers invariably bend by a small amount, thus changing their apparent location in the x-y plane. This re-arrangement of fibers is revealed by the clustering seen in Fig. 3(a). In light of this problem, an alternate approach was adopted. Notably, the average fiber extension, $\bar{\Delta}$, was calculated as the difference between the average height of the exposed fibers and the average height of the reference plane within the dissolved section. This calculation yielded an average extension $\bar{\Delta} = 76$ μm, with a corresponding strain [following equation (1)] of $\varepsilon_i = -0.152\%$.

The distribution of fiber heights above the average value within the reference plane is plotted in Fig. 3(c). Also shown for comparison is a calculated distribution, based upon the assumption that each of the
exposed fibers has extended by the same amount (equal to the average value, $\bar{\Delta} = 76 \mu m$) and that the distribution in apparent heights is associated only with the height variation of the fibers prior to matrix dissolution, i.e. the height distribution of the reference plane. The agreement between the calculated and measured distribution supports the idea that each of the fibers did indeed extend by the same amount (within $\sim 1 \mu m$) and that the apparent height distribution is due solely to the tilt of the reference plane with respect to the beam axis.

The measurements for panels B and C are shown in Figs 4 and 5, respectively. The average fiber extensions for the two panels are $94$ and $112 \mu m$, with
corresponding strains, $\varepsilon = -0.188\%$ and $-0.207\%$.
The calculated apparent height distribution for panel C (the one with the thicker cladding) is in good agreement with the measured distribution [Fig. 4(b)].
The correlation between the two distributions for panel B is somewhat worse, with the measured distribution being noticeably broader than the calculated one, by $\sim 10 \mu m$ [Fig. 5(b)].

The discrepancy between the calculated and measured distributions in Fig. 5(b) is believed to be due to the "splaying" of the fibers following dissolution, causing an apparent reduction in fiber extension. This effect is shown schematically in Fig. 6. Assuming that the fiber is initially straight and parallel to the beam axis, the apparent foreshortening of the fiber, $\delta\varepsilon$, following dissolution is given approximately by

$$\delta\varepsilon \approx \frac{r}{2L},$$

where $r$ is the distance through which the top of the fiber has been displaced. In general, all the fibers are not displaced by the same amount and thus the apparent foreshortening varies among the fibers. Consequently, the measured height distribution is broadened.

The effects of fiber displacement (in the $x$-$y$ plane) on the apparent height distribution have been incorporated into the analysis in an approximate manner. For this purpose, it should be noted that there is no information regarding the displacement of any individual fiber, only information regarding the distribution of locations in relation to the initial distribution. These displacements can be considered to occur in two steps. In the first, the fibers are displaced in the same direction by a fixed amount, $r$. This step is equivalent to shifting the centroid of the fiber distribution in the $x$-$y$ plane by $r_c$. In the second step, the fibers are allowed to splay about their centroid. The first of these two steps gives rise to a systematic error in the measured fiber heights, by an amount

$$\delta\Delta \approx \frac{r^2}{2L}. \quad (3)$$

This term must be added to the apparent average fiber extension, $\Delta\varepsilon$. The second step gives rise to either a broadening or narrowing of the height distribution by an amount $\Delta r$, depending on whether the fibers become more dispersed or clustered together; it may also give rise to a systematic error.

Figure 7 shows the spatial distribution of the fibers after matrix dissolution within the $x$-$y$ plane. The results are presented in terms of the distances between the fibers after dissolution and their centroid $(x_c, y_c)$. The shift in the centroid following dissolution is given by

$$r_c = \sqrt{x_c^2 + y_c^2}. \quad (4)$$

Also shown for comparison are the calculated fiber distributions about the initial centroid, $(0, 0)$, prior to dissolution, assuming that each location within the dissolved section is an equally probable site for a fiber. The magnitude of the centroid shift, $r_c$, provides
is obtained by assuming that the outermost fiber in the initial distribution is responsible for the outermost fiber after dissolution. The expected broadening in the fiber height distribution is thus given approximately by

$$\delta \Delta_h \approx \frac{(r_1 + \Delta r)^2 - r_1^2}{2l}.$$  

(a) Before Dissolution

(b) After Dissolution

Fig. 2. Schematics showing (a) specimen geometry used to measure the residual fiber strain, and (b) extension of exposed fibers following matrix dissolution.

a measure of the systematic shift in the fiber bundle, whereas the differences between the measured and calculated distributions about the relevant centroid provide a measure of the degree of splaying and thus the broadening or narrowing of the measured height distribution.

The measured distributions for panels A and C [Figs 7(a) and (c)] are consistent with the calculated ones, suggesting that there is a little change in the width of the fiber height distributions associated with fiber splaying. Indeed, the correlations between the measured and calculated height distributions for the two panels [Figs 3(c) and 5(c)] indicate that the fiber height distribution is associated solely with the height distribution in the reference plane: the splaying effect is negligible. The shifts in the centroids of the fiber bundles are \( \sim 0.13 \) and \( 0.9 \) mm for panels A and C, respectively; the corresponding corrections in fiber heights [following equation (3)] are \( \sim 0.2 \) and \( \sim 8 \) \( \mu \)m. The correction for panel A is negligible in comparison with the precision of the measurements, whereas the correction for panel C is substantially larger. These corrections are summarized in Table 2.

For panel B, the centroid shift is \( r_c = 0.4 \) mm. The correction for the fiber heights is \( \delta \Delta \sim 1.6 \) \( \mu \)m (Table 2). However, the spatial distribution of fibers is substantially broader than the calculated one: \( \sim 1 \) vs \( \sim 1.6 \) mm. A conservative estimate of the associated broadening in the fiber height distribution

Fig. 3. Panel A: (a) a map of the locations of the measurements both within the intact portion of the composites (shaded) and within the exposed region (between the shaded regions); (b) distribution in differences between the measured fiber heights on the reference plane and the ones obtained from a least squares fit; and (c) distribution in fiber heights above the average value in the reference plane within the dissolved section of the composite.
Taking $\Delta r = 0.6 \text{ mm}$, $r_c = 0.4 \text{ mm}$ and $l = 30 \text{ mm}$ in equation (5) yields the result $\delta \Delta \approx 8 \mu m$. This broadening is of the same order as the one measured experimentally [Fig. 7(b)]. At this point, it is not clear how the broadening effect influences the average fiber extension. Such effects have been neglected in computing the values presented in Table 2.

4.2. Residual stresses

The residual stresses corresponding to the measured (corrected) fiber strains have been calculated using the concentric cylinder model [13]. For a prescribed thermal strain

$$\Omega = (\alpha_f - \alpha_m)\Delta T,$$

the axial stresses, $\sigma_i$ and $\sigma_m$, in the fibers and the matrix are given by

$$\sigma_i = -(1 - f) \frac{E_i E_m (E_i + E_m)}{E_i (1 - 2v_i) + E_m (1 - 2v_m)} \Omega$$

Fig. 4. Panel B: (a) a map of the locations of the measurements both within the intact portion of the composites (shaded) and within the exposed region (between the shaded regions); (b) distribution in differences between the measured fiber heights on the reference plane and the ones obtained from a least squares fit; and (c) distribution in fiber heights above the average value in the reference plane within the dissolved section of the composite.

Fig. 5. Panel C: (a) a map of the locations of the measurements both within the intact portion of the composites (shaded) and within the exposed region (between the shaded regions); (b) distribution in differences between the measured fiber heights on the reference plane and the ones obtained from a least squares fit; and (c) distribution in fiber heights above the average value in the reference plane within the dissolved section of the composite.
where $E$ is the longitudinal composite modulus \( [fE_f + (1 - f)E_m] \) and $v$ is Poisson's ratio (assumed to be the same for the fibers and the matrix). The radial and hoop stresses acting on the fiber are

\[
\sigma_r = \sigma_t = \frac{(1 - f)E_m E_f}{E_f + E(1 - 2v)} \Omega. \tag{9}
\]

Alternatively, the stresses, $\sigma_t$ and $\sigma_m$, and the misfit strain, $\Omega$, can be written in terms of the axial fiber strain, $\varepsilon_f$, through the relations

\[
\sigma_t = \frac{E(E_t + E)}{E_t + E(1 - 2v)} \varepsilon_f \tag{10}
\]

\[
\sigma_m = -\frac{fE(E_t + E)}{(1 - f)(E_t + E(1 - 2v))} \varepsilon_f \tag{11}
\]

\[
\Omega = -\frac{E}{(1 - f)E_m} \varepsilon_f. \tag{12}
\]

The foregoing analysis is based upon the assumption that the fibers are distributed uniformly throughout the composite. For the clad structures, the same results for $\sigma_t$ and $\sigma_m$ are obtained by considering the panel to be comprised of three laminae (two of monolithic Ti and one of a fiber-reinforced TMC with a volume fraction, $f_c = 0.32$) and invoking the usual compatibility and equilibrium conditions (see the Appendix).

The values of the thermal strain, $\Omega$, inferred from equation (12), are summarized in Table 2. The relevant constituent properties are summarized in Table 3. The matrix modulus, $E_m$, is taken to be the average value over the temperature range, 20–650°C. The values of $\Omega$ obtained for the two panels reinforced with the SCS-6 fibers are almost equivalent to one another (0.420 vs 0.412%). Indeed, since the two panels are comprised of the same constituents, the thermal strain is expected to be the same. The thermal strain for the panel containing Sigma fibers is slightly higher (0.435%), by $\sim 5\%$. This result is consistent with the differences in thermal expansion coefficients of the two fibers (Table 3). The values of $\Omega$ have been combined with equation (6) and the thermal expansion coefficients of the matrix and fibers to infer the relevant temperature change, $\Delta T$, associated with the residual strain. The values are consistently in the range $[\Delta T] \approx 750$–1000°C.† These results are consistent with the low creep resistance of the matrix in this temperature range, causing thermal strains at yet higher temperatures to be relaxed during cooling from the processing temperature.

The calculated residual stresses are presented in Table 4 and Fig. 8. Also shown in Fig. 8 are the calculated curves based on an average thermal expansion coefficient of the matrix and fibers to infer the relevant temperature change, $\Delta T$, associated with the residual strain. The values are consistently in the range $[\Delta T] \approx 750$–1000°C.† These results are consistent with the low creep resistance of the matrix in this temperature range, causing thermal strains at yet higher temperatures to be relaxed during cooling from the processing temperature.

Fig. 7. Distributions in fiber locations with respect to the centroid of the dissolved sections: (a) panel A, (b) panel B and (c) panel C.

†The range of temperature is associated with the range of thermal expansion coefficients reported for the Ti-6Al-4V alloy [14].
strain $\Omega \approx 0.42\%$ and the two fiber moduli (360 and 410 GPa) coupled with the solutions for the concentric cylinder model.

5. DISCUSSION

The present method of measuring residual fiber strains has several potential advantages over the selective etching method described in Ref. [1]. In the latter method, all of the matrix within a central portion of a strip of composite is dissolved and the fiber strain obtained from the average extension of the composite. The extension is measured by first placing indents or other fiducial points on the ends of the specimen and on the ends of a reference bar and measuring the distances between corresponding pairs of fiducial points both before and after dissolution. The measurements are made using optical micrographs, typically taken at a magnification of $\sim 200 \times$. The expected precision of an individual measurement is $\sim 1-2 \, \mu\text{m}$ (assuming that the precision of the measurement on the micrograph is $\sim 0.2-0.4 \, \text{mm}$). Four such measurements are required to obtain the length change: two before dissolution and two after. The precision in the extension is thus $\sim 2-4 \, \mu\text{m}$ (taking the root of the sum of the squares of the precision of the individual measurements). Assuming that the precision in the computed fiber strain is dominated by the length change measurement (and not by the measurement of gauge length), and using typical values of gauge length ($\sim 50 \, \text{mm}$) and residual fiber strain ($\sim 0.2\%$), the resulting precision is $\sim 2-4\%$ of the fiber strain. In the method described in this study, the precision of individual measurements was $\sim 1 \, \mu\text{m}$. Since the determination of length change requires two such measurements (assuming that the top surface is normal to the beam direction so that the tilt corrections can be neglected), the precision in length change is $\sim 1.4 \, \mu\text{m}$. Using the values of gauge length and fiber strain quoted above, the corresponding precision in the fiber strain is $\sim 1.4\%$; somewhat smaller than that obtained using optical micrographs. However, this estimate is conservative, since many measurements are made on each sample and the average extension used to calculate the fiber strain. Assuming that each fiber extends by the same amount, the precision of the average extension is expected to be reduced by a factor of $1/\sqrt{N}$, where $N$ is the number of measurements.

It is anticipated that the resolution of the present method could be improved further in one of three ways. In the first, the surfaces on which the measurements are made could be polished prior to dissolution. In the present experiments, these surfaces had been prepared by EDM and (as noted earlier) the precision of individual measurements was $\sim 1 \, \mu\text{m}$. With the improvements in surface condition, it is expected that the precision could approach the resolution of the confocal microscope (0.2 $\mu\text{m}$). The second improvement involves the splaying that occurs after dissolution and the difficulty in identifying the locations of the fibers prior to dissolution. The locations of the fibers could be marked prior to dissolution by putting indentations on the top surfaces of the fibers and mapping out their locations in the $(x, y)$ plane both before and after dissolution. Consequently, the extension of each individual fiber could be obtained directly without the need for the splaying corrections introduced here. Finally, the precision of the strains could be improved by increasing the gauge length of the exposed fibers, as dictated by the length of the slits. This length may be limited by the size of the panel or by the space available in the microscope.

In the present method, the fiber strains are fully relaxed after dissolution, regardless of the degree of uniformity in the strains. In contrast, in the other method, fibers with a higher than average residual

<table>
<thead>
<tr>
<th>Panel</th>
<th>Uncorrected fiber strain, $c^u$</th>
<th>Corrected fiber strain, $c^c$</th>
<th>Thermal strain, $\Omega$</th>
<th>Temperature range, $[\Delta T]$ (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$-0.152%$</td>
<td>$-0.152%$</td>
<td>$0.435%$</td>
<td>$750-990$</td>
</tr>
<tr>
<td>B</td>
<td>$-0.188%$</td>
<td>$-0.191%$</td>
<td>$0.420%$</td>
<td>$760-1020$</td>
</tr>
<tr>
<td>C</td>
<td>$-0.207%$</td>
<td>$-0.222%$</td>
<td>$0.412%$</td>
<td>$750-1000$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Constituent</th>
<th>Young's modulus, $E$ (GPa)</th>
<th>Thermal expansion coefficient, $\alpha$ (K$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCS-6 SiC</td>
<td>410 (Provided by fiber manufacturer)</td>
<td>$4.9 \times 10^{-6}$ (from Ref. [16])</td>
</tr>
<tr>
<td>Sigma SiC</td>
<td>360 (Measured on extracted fibers, using a laser extensometer to measure strains)</td>
<td>$4.6 \times 10^{-6}$ (Provided by fiber manufacturer)</td>
</tr>
<tr>
<td>Ti-6Al-4V</td>
<td>114 (At 20$^\circ$C [15])</td>
<td>$(9.0-10.4) \times 10^{-6}$ (Average values, 20-650$^\circ$C [14])</td>
</tr>
<tr>
<td></td>
<td>91 (Average value, 20-650$^\circ$C [15])</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Residual strains

Table 3. Thermoelastic properties of the constituents
with the point at which the matrix creeps rapidly. It is expected that the method could be applied to other metal or intermetallic matrix composite reinforced with large diameter (monofilament) fibers, provided the matrix can be selectively etched.

A rudimentary analysis of the effects of fiber displacement following matrix dissolution on the fiber extension has been presented. In some cases, these effects are negligibly small in comparison with the precision of the individual measurements; in others, they are somewhat larger, though they are still \( \leq 5 \) \% of the measured fiber extensions.

The present method also has advantages over the X-ray and neutron diffraction techniques in terms of the simplicity of the test procedure as well as the interpretation of the measurements.

### 6. CONCLUDING REMARKS

A new method has been developed to measure the residual fiber strains in fiber-reinforced Ti matrix composites. The method has been demonstrated on a series of Ti/SiC composites with varying fiber content. The fiber strains have been used to calculate the fiber and matrix stresses through a concentric cylinder model. The results are consistent with the thermoelastic properties of the constituents and an assumed maximum temperature that is consistent

### REFERENCES


### APPENDIX

#### Axial Fiber Strain in Clad Structures

One approach to interpreting the residual fiber strain in a clad TMC is to assume that the fibers are distributed
uniformly through the composite and subsequently apply
the analysis of the concentric cylinder model. The resultant
relationship between the misfit strain, $\Omega$ [defined in equation
(6)], and the fiber strain, $\epsilon_f$, is [from equation (12)]

$$\epsilon_f = -\frac{(1 - f)E_n}{E} \Omega,$$  \hspace{1cm} (A1)

where $f$ is the overall fiber volume fraction

$$f = \frac{t_f}{t_n + t_f}$$  \hspace{1cm} (A2)

with $2t_n$ being the thickness of the (inner) reinforced, $t_0$ the thickness of the cladding and $f$, the fiber volume fraction
within the reinforced region. The longitudinal
modulus, $E$, of the structure is thus given by

$$E = fE_f + (1 - f)E_m$$  \hspace{1cm} (A3)

$$= \frac{fE_f}{1 + f_0/t_f} + \left(1 - \frac{f_0}{1 + f_0/t_f}\right)E_m.$$  \hspace{1cm} (A4)

Combining equations (A1)-(A4) yields the result

$$\epsilon_f = -\frac{(1 - f_0 + t_0/t_f)E_n}{(1 - f_0 + t_0/t_f)E_m + f_0E_f} \Omega,$$  \hspace{1cm} (A5)

Alternatively, the clad structure can be considered to be
a laminate comprised of one layer of a fiber-reinforced TMC
with a fiber volume fraction, $f_0$, and two monolithic Ti
layers. The residual fiber strain can then be partitioned into
two components: one due to the thermal mismatch between
the fibers and matrix within the TMC lamina (in the absence
of cladding) and another due to the mismatch between
the TMC and the Ti laminae. The former contribution,
following equation (12), is

$$\epsilon_f = -\frac{(1 - f_0)E_n}{(1 - f_0)E_m + f_0E_f} \Omega.$$  \hspace{1cm} (A6)

The latter contribution is evaluated by invoking the usual
conditions of compatibility and mechanical equilibrium
during cooling. The condition for compatibility along the
fiber direction can be written as

$$\frac{\sigma_{AI}}{E_{AI}} + \frac{\sigma_{AI}}{E_{AI}} \Delta T = \frac{\sigma_{BI}}{E_B} = \frac{\sigma_{BI}}{E_B} + \sigma_{BI} \Delta T,$$  \hspace{1cm} (A7)

where the subscript A refers to the TMC lamina; B refers
to the Ti cladding; 1 and 2 are the directions parallel and
perpendicular to the fiber axis, respectively; and $\nu_2$ is
the Poisson’s ratio of the TMC associated with loading in
the fiber direction. Since the cladding material is the same
as the matrix within the TMC, it follows that $\sigma_B = \sigma_2$,
$\nu_B = \nu_0$ and $E_B = E_m$. The fiber and matrix are assumed
to have the same Poisson’s ratio; consequently $\nu_2 = \nu_0$. The
longitudinal thermal expansion coefficient of the TMC is
given by

$$\alpha_{AI} = \alpha_0 + \frac{f_0E_f(\alpha_f - \alpha_0)}{f_0E_f + (1 - f_0)E_f}.$$  \hspace{1cm} (A8)

The condition for equilibrium in the fiber direction is

$$\sigma_{AI} + \sigma_{BI} = 0.$$  \hspace{1cm} (A9)

Combining equations (A7)-(A9) yields the axial strain
associated with the ply-level stress. When this term is added
onto the residual strain due to the fiber/matrix mismatch
[equation (A6)], the result is equivalent to the one given in
equation (A5).